Equality of directional critical exponents in multiparameter percolation models

1994 J. Phys. A: Math. Gen. 27 1851

(http://iopscience.iop.org/0305-4470/27/6/012)

J. Phys. A: Math. Gen. 27 (1994) 1851–1858. Printed in the UK

Equality of directional critical exponents in multiparameter percolation models

John C Wierman

Mathematical Sciences Department, The Johns Hopkins University, Baltimore, MD 21218, USA

Received 2 August 1993

Abstract. In multiparameter percolation models, a directional critical exponent may be considered for each direction of approach in the parameter space to a point on the critical surface. An elementary coupling argument shows that for each point on the critical surface the directional critical exponents are equal for all directions strictly in the positive orthant. Using the substitution method, the set of directions with equal directional critical exponents can be extended beyond the orthant. The anisotropic triangular lattice bond percolation model is used to illustrate the method.

1. Introduction

Several forms of multiparameter percolation models have been investigated. Anisotropic bond percolation models with different occupancy probabilities for bonds or sites in different directions were studied by Sykes and Essam (1964) and Kesten (1982). Mixed models, having different parameters associated with sites and bonds, were studied by Hammersley (1980), McDiarmid (1980, 1981) and Wierman (1984).

General percolation models have been investigated from a variety of perspectives. Books devoted to the subject are a mathematical monograph focusing on two-dimensional models by Kesten (1982), a mathematical discussion of results for Z^d emphasizing higher dimensions by Grimmett (1989), an elementary description of physical applications by Efros (1986), a discussion of scaling theory and numerical methods by Stauffer (1985), and Durrett (1988), which describes the connections between interacting particle systems and percolation models. Recent reviews of Aizenman (1987), Kesten (1987), and Newman (1987) concern critical exponents and scaling theory for percolation models.

A multiparameter percolation model with parameters $p_1, p_2, \ldots, p_d$ has parameter space $[0, 1]^d$. The percolative region $\mathcal{P}$ is the set of all parameter values for which infinite open clusters exist with positive probability. The boundary of $\mathcal{P}$ is called the critical surface.

The exact critical surface is known only in special cases. For the square lattice bond model with parameter p for horizontal bonds and q for vertical bonds, the critical surface is $p + q = 1$. The three-parameter bond model on the triangular lattice with parameters u, v, and w has critical surface $1 - u - v - w + uvw = 0$. Both of these solutions were discovered by Sykes and Essam (1964). The anisotropic square lattice solution, as well as a two-parameter anisotropic triangular lattice model, was rigorously verified by Kesten (1982). Proof of the full three-parameter anisotropic triangular lattice solution is implied by Kesten (1982), Menshikov *et al* (1986), and Aizenman and Barsky (1987), establishing the equivalence of various versions of the critical probability.

0305-4470/94/061851+08$19.50

In one-parameter percolation models, behaviour near the critical probability is described in terms of critical exponents. For multiparameter models, a point on the critical surface may be approached along a variety of directions, and an appropriate critical exponent may be defined for each direction. It is generally believed that the critical exponents corresponding to all directions of approach from within $\mathcal{P}$ should be equal, although this has not yet been proved. This note provides some progress toward a proof.

The points in the parameter space $[0,1]^d$ may be partially ordered by letting $p = (p_1, p_2, \ldots, p_d) \leqslant q = (q_1, q_2, \ldots, q_d)$ if and only if $p_i \leqslant q_i$ for all $i = 1, 2, \ldots, d$. In section 2, an elementary coupling argument proves that in any d-parameter percolation model, the critical exponents β and γ for approaching a point p^* on the critical surface are equal for all directions of approach in the orthants $\{p : p^* < p\} \cap [0,1]^d$ and $\{p : p^* > p\} \cap [0,1]^d$, respectively. The substitution method is adapted to the problem in section 3. The method has led to considerable improvement of the rigorous critical probability bounds for the Kagomé lattice bond model (Wierman 1990), and a proof of equality of the critical exponents of the triangular and hexagonal lattice bond percolation models (Wierman 1991). Using the substitution method, directions of approach to the critical surface outside the orthants are considered for the three-parameter triangular lattice bond model in section 4. The substitution method uses the detailed structure of the lattices to compare the behaviour of the percolation model at two different parameter values.

2. General multiparameter models

In this section, we consider a multiparameter Bernoulli percolation model on a connected infinite graph G. That is, for some integer $d \geqslant 2$ there are d classes $\mathcal{E}_i$ of elements (which may be edges or vertices) such that each element of $\mathcal{E}_i$ is open with probability p_i, $0 \leqslant p_i \leqslant 1$, independently of all other elements of G.

For any vertex v, let C_v denote the open cluster containing v, i.e. the set of elements of G which may be reached through open paths from v. The *percolation probability* function $\theta_v(p)$, $p \in [0,1]^d$, represents the probability that v is in an infinite open cluster:

$$\theta_v(p) = P_p[|C_v| = \infty].$$

The *mean cluster size* is defined by

$$\chi_v(p) = E_p|C_v|.$$

Since G is connected, for each p, either $\theta_v(p) = 0$ for all v or $\theta_v(p) > 0$ for all v. Similarly, $\chi_v(p) < +\infty$ for all v or $\chi_v(p) = +\infty$ for all v. Thus, we define *percolative regions* corresponding to these criteria (omitting the dependence on v from the notation):

$$\mathcal{P}_H = \{p \in [0,1]^d \quad : \quad \theta(p) > 0\}$$

and

$$\mathcal{P}_T = \{p \in [0,1]^d \quad : \quad \chi(p) < +\infty\}.$$

For a set $A \subset [0,1]^d$, we say $x \in \partial A$ if every neighborhood of x contains an element of A and an element of $[0,1]^d - A$. The *critical surfaces* are then defined by $\mathcal{S}_H = \partial\mathcal{P}_H$ and $\mathcal{S}_T = \partial\mathcal{P}_T$.

For a point p on critical surface $\mathcal{S}_H$, and a direction given by vector t, we define the critical exponent $\beta(t)$ by

$$\beta(t) = \lim_{\epsilon \downarrow 0} \frac{\log[\theta(p + \epsilon t) - \theta(p)]}{\log \epsilon \|t\|}$$

where $\|\cdot\|$ denotes Euclidean distance. If the limit does not exist, then denote the lim sup and lim inf by superscripts $+$ and $-$ (respectively) on β. Similarly, for $p \in \mathcal{S}_T$ and a direction t, define

$$\gamma(t) = \lim_{\epsilon \downarrow 0} \frac{\log \chi(p - \epsilon t)}{\log \epsilon \|t\|}.$$

We now show that there is a common value of $\beta(t)$ for all $t > 0$ componentwise. A similar argument will establish the result for $\gamma(t)$.

First, note that a standard coupling argument establishes that if $p \leqslant q$ then $\theta(p) \leqslant \theta(q)$. Let $\{u_e, e \in G\}$ be independent identically distributed uniform (0, 1) random variables, each associated with an element e of G. Construct a percolation model with paramenter p by declaring each element $e \in \mathcal{E}_i$ to be open if $u_e \leqslant p_i$. A model with parameter q may be constructed similarly using the same random variables $\{u_e\}$. However, any element that is open in the first model is also open in the second model. Therefore, the existence of an infinite open cluster in the first model implies existence of an infinite open cluster in the second model, so $\theta(p) \leqslant \theta(q)$.

Now, let $p \in \mathcal{S}_H$, and $s > 0$ and $t > 0$ be different direction vectors in the parameter space. Let $M = \max\{t_i/s_i\}$ and $m = \min\{t_i/s_i\}$, so $ms \leqslant t \leqslant Ms$. Then, for $\epsilon > 0$,

$$p + \epsilon ms \leqslant p + \epsilon t \leqslant p + \epsilon Ms$$

which, by the previous argument, implies that

$$\theta(p + \epsilon ms) \leqslant \theta(p + \epsilon t) \leqslant \theta(p + \epsilon Ms).$$

Taking logarithms and dividing by $\log \|\epsilon t\|$ (which is negative for ϵ sufficiently small) yields

$$\frac{\log[\theta(p + \epsilon ms) - \theta(p)]}{\log \|\epsilon t\|} \geqslant \frac{\log[\theta(p + \epsilon t) - \theta(p)]}{\log \|\epsilon t\|} \geqslant \frac{\log[\theta(p + \epsilon Ms) - \theta(p)]}{\log \|\epsilon t\|}$$

which implies that

$$\frac{\log[\theta(p + \epsilon ms) - \theta(p)]}{\log \|\epsilon ms\|} \geqslant \frac{\log[\theta(p + \epsilon t) - \theta(p)]}{\log \|\epsilon t\|} \geqslant \frac{\log[\theta(p + \epsilon Ms) - \theta(p)]}{\log \|\epsilon Ms\|}.$$

Letting $\epsilon \to 0$, if the limits defining $\beta(t)$ and $\beta(s)$ exist, we have

$$\beta(s) = \beta(t).$$

If the defining limits do not exist, then $\beta^+(s) = \beta^+(t)$ and $\beta^-(s) = \beta^-(t)$.

3. Substitution method

The substitution method was introduced by Wierman (1990) for the purpose of deriving bounds for critical probabilities of bond percolation models. However, the underlying idea of the method is the comparison of the behaviour of two different percolation models, which is useful in the study of critical exponents as well.

To describe the application of the substitution method in a bond percolation model (for convenience), consider a lattice $\mathcal{G}$ which may be decomposed into a union of isomorphic finite edge-disjoint connected subgraphs, in such a way that every edge is in a subgraph and every vertex is in at least one subgraph. Vertices which are in more than one subgraph are called *boundary vertices*. A path on the lattice may enter or leave a subgraph in the decomposition only through its boundary vertices.

Consider a fixed subgraph G in this decomposition. Denote its boundary vertices by $A_1, A_2, \ldots, A_k$ for some k. Any configuration (i.e. designation of bonds as open or closed) on G determines a partition of the boundary vertices into clusters of vertices which are connected by open bonds. Each such *boundary partition* may be denoted by a sequence of vertices and vertical bars, where vertices are in distinct open clusters if and only if they are separated by a vertical bar. For example, if the boundary vertex set consists of three vertices, A_1, A_2, and A_3, then $A_1A_2|A_3$ indicates that A_1 and A_2 are in the same open cluster but A_3 is in a different open cluster.

The bond percolation model on $\mathcal{G}$ with parameter p assigns a probability to each configuration on G. A probability $P_p^{\mathcal{G}}(\pi)$ for each boundary partition π is determined by summing the probabilities of all configurations which produce the partition π.

The set of boundary partitions is a partially ordered set. A partition π is a *refinement* of a partition σ, denoted $\pi \leqslant \sigma$, if every cluster of π is contained entirely in a cluster of σ, or, equivalently, every cluster of σ decomposes into clusters of π. The set of boundary partitions ordered by refinement is a combinatorial lattice called the *partition lattice*.

Suppose that another lattice $\mathcal{H}$ may be decomposed into subgraphs with k boundary vertices, corresponding to the substitution of a subgraph for each subgraph in the decomposition of $\mathcal{G}$. Consider the subgraph H in $\mathcal{H}$ substituted for G in $\mathcal{G}$, and identify the boundary vertices with $A_1, A_2, \ldots, A_k$. As above, a probability measure $P_q^{\mathcal{H}}$ on the partition lattice of partitions of $\{A_1, A_2, \ldots, A_k\}$ is determined, from the percolation model on $\mathcal{H}$ with parameter q.

In applying the substitution method to percolation models, the relevant comparison of two probability measures $P_p^{\mathcal{G}}$ and $P_q^{\mathcal{H}}$ on the partition lattice is *stochastic ordering*: a *filter* in a partially ordered set S is a subset $F \subset S$ such that if $g \geqslant f$ and $f \in F$, then $g \in F$. If P and Q are two probability measures on S, then we say that P is *stochastically smaller* than Q, denoted $P \leqslant_S Q$, if $P[F] \leqslant Q[F]$ for every filter F. The set of probability measures on S are partially ordered by stochastic ordering.

Let $\mathcal{B}$ denote the union of the sets of boundary vertices of subgraphs in the decompositions of $\mathcal{G}$ and $\mathcal{H}$ (with identification of boundary vertices of corresponding subgraphs in $\mathcal{G}$ and $\mathcal{H}$). A result of Preston (1974) implies that if $P_p^{\mathcal{G}} \leqslant_S P_q^{\mathcal{H}}$, then there is a coupling of the percolation models on each pair of corresponding subgraphs G and H, in which $C_v \cap \mathcal{B}$ on G with parameter p is contained in $C_v \cap \mathcal{B}$ on H with parameter q. By edge-disjointness, these couplings together give a coupling of the percolation models in which $C_v \cap \mathcal{B}$ in $\mathcal{G}$ with parameter p is contained in $C_v \cap \mathcal{B}$ in $\mathcal{H}$ with parameter q. Therefore, $\theta^{\mathcal{G}}(p) \leqslant \theta^{\mathcal{H}}(q)$ and $\chi^{\mathcal{G}}(p) \leqslant \chi^{\mathcal{H}}(q)$.

In this paper, we wish to apply the method to two families of probability distributions arising from approaching a point on the critical surface of a multiparameter bond model

in two different directions. Letting $p_0 \in S_H$, and t_1 and t_2 be unit direction vectors, we wish to compare $P^{\mathcal{G}}_{p_0+\epsilon t_1}$ and $P^{\mathcal{G}}_{p_0+\delta t_2}$ via stochastic ordering. This will be accomplished using directional derivatives of the filter probability functions, as described in the following paragraphs.

Note that, for a fixed decomposition of $\mathcal{G}$ into isomorphic subgraphs, there are a finite number of filter probability functions, each of which is a polynomial function of the parameters. Therefore, the filter probabilities are differentiable with respect to each parameter, and a gradient vector exists for each filter probability. Directional derivatives may be computed by taking the inner product of the direction vector with the gradient.

Suppose that for two directions, t_1 and t_2, the directional derivatives of all the filter probabilities at p_0 are positive. Then there exist constants c and C such that for every filter F

$$P^{\mathcal{G}}_{p_0+c\epsilon t_2}[F] \leqslant P^{\mathcal{G}}_{p_0+\epsilon t_1}[F] \leqslant P^{\mathcal{G}}_{p_0+C\epsilon t_2}[F]$$

for $\epsilon > 0$ sufficiently small. Thus

$$\theta^{\mathcal{G}}(p_0 + c\epsilon t_2) \leqslant \theta^{\mathcal{G}}(p_0 + \epsilon t_1) \leqslant \theta^{\mathcal{G}}(p_0 + C\epsilon t_2)$$

which, as in section 2, leads to

$$\beta(t_1) = \beta(t_2)$$

if the defining limit exists in either direction, and $\beta^+(t_1) = \beta^+(t_2)$ and $\beta^-(t_1) = \beta^-(t_2)$ otherwise.

4. Triangular lattice model

To illustrate the application of the substitution method, consider the three-parameter triangular lattice bond model, in which bonds are open with probability p, q, or r depending on the direction of the bond. To apply the substitution method, partition the lattice into edge disjoint triangles. Label the vertices of one such triangle A, B, and C, so that edge AC has parameter p, AB has parameter q, and BC has parameter r.

The probabilities of the four non-minimal boundary vertex partitions are

$$P[ABC] = pqr + pq(1-r) + qr(1-p) + pr(1-q)$$

$$P[AB|C] = q(1-p)(1-r) \qquad P[AC|B] = p(1-q)(1-r) \qquad P[BC|A] = r(1-p)(1-q)$$

and the eight filter probabilities are of the form

$$pqr+pq(1-r)+qr(1-p)+pr(1-q)+iq(1-p)(1-r)+jp(1-q)(1-r)+kr(1-p)(1-q)$$

for $i = 0, 1$; $j = 0, 1$; and $k = 0, 1$.

Thus, the gradients of the filter probabilities, written as column vectors, have the form:

$$\begin{pmatrix} q+r-2qr-i(q-qr)+j(1-q-r+qr)-k(r-qr) \\ p+r-2rp+i(1-q-r+qr)-j(p-pr)-k(r-pr) \\ p+q+2pq-i(q-pq)-j(p-pq)+k(1-p-q-pq) \end{pmatrix}.$$

As discussed in section 3, any direction which has a positive component in the direction of all eight gradients will have the same β critical exponent value as any direction in the positive orthant. Equivalently, any vector lying 'above' all eight half-planes through the origin which are orthogonal to these gradient vectors has the same β and γ values as directions in the non-negative orthant. The eight gradient vectors are

$$(q+r-2qr,\ p+r-2pr,\ p+q-2pq) \qquad (r-qr,\ 1-pr,\ p-pq)$$
$$(1-qr,\ r-pr,\ 1-pq) \qquad (q-qr,\ p-pr,\ 1-qp)$$
$$(1-q,\ 1-p,\ 0) \qquad (0,\ 1-r,\ 1-q)$$
$$(1-r,\ 0,\ 1-p) \qquad (1-q-r+qr,\ 1-p-r+pr,\ 1-p-q+pq)\,.$$

Note that if a direction vector t has a strictly positive inner product with all 8 vectors, then (by viewing the inner product as a continuous function of the coordinates of the direction vector) every direction in a neighbourhood of t does also. Since all the non-zero components of these gradient vectors are positive, for $a > 0$ and $b > 0$, vectors of the forms $(0, a, b)$, $(a, 0, b)$, and $(a, b, 0)$ all satisfy this condition. Therefore, for ϵ sufficiently small, directions of the form $(-\epsilon, a, b)$, $(a, -\epsilon, b)$, and $(a, b, -\epsilon)$ have the same β and γ values as directions in the positive orthant. Hence, the set of such directions extends beyond the positive orthant for this multiparameter model.

As a special case, we consider the point on the critical surface with all coordinates equal: $p = q = r = 2\sin(\pi/18)$. In this case, the gradient vectors (rescaled) give the directions

$$(1,1,1) \qquad (a,1,a) \qquad (1,a,a) \qquad (a,a,1)$$
$$(1,1,0) \qquad (0,1,1) \qquad (1,0,1) \qquad (1,1,1)$$

where $a = 2\sin(\pi/18) - \sin(\pi/18)^2 \approx 0.2267$. The direction $(1, 1, 1)$ is normal to the critical surface at this point, so allows any direction of approach from within the percolative region. The region satisfying all six remaining constraints is the intersection of half-spaces, and thus is convex. The region may be characterized as the convex combinations of multiples of the six rays

$$\left(1,-1,\frac{1-a}{a}\right) \qquad \left(1,\frac{1-a}{a},-1\right) \qquad \left(-1,1,\frac{1-a}{a}\right)$$
$$\left(\frac{1-a}{a},1,-1\right) \qquad \left(-1,\frac{1-a}{a},1\right) \qquad \left(\frac{1-a}{a},-1,1\right)$$

which are the intersections of pairs of planes perpendicular to the remaining gradient vectors.

5. Discussion and remarks

While the substitution method calculations explicitly used the detailed structure of the lattice through its use of filter probabilities, it made use of the exact critical surface only in characterizing the solution region: the positivity of components of the gradient vectors was valid for any $(p, q, r) \in (0, 1)^3$, and thus, in particular, for any point (p, q, r) on the critical surface. Thus, the method may show that the solution region extends beyond the

positive orthant for unsolved multiparameter models and, in particular, is not restricted to two-dimensional lattices only.

The two-parameter square lattice bond percolation model, with different parameters for the horizontal and vertical bonds, was also considered. The substitution method was initially applied using two different regions for the unit of substitution: a four-pointed star, and a single square. In both cases, two filters caused restriction to the positive quadrant. For the four-pointed star, the filter corresponded to the events that the opposite vertices were connected by an open path, for which the probability is a function of only one parameter in each case. For the square, the filters also corresponded to events with probabilities independent of one of the parameters. By considering larger subgraphs of the square lattice, the solution region may be extended beyond the quadrant, since none of the filter events will have probabilities independent of a parameter.

A larger subgraph of the triangular lattice may also be considered. As with the square lattice, it is expected that the solution region will become larger as the substituted region is enlarged. This is suggested by a computational method currently under investigation. It is an open question whether the solution region converges to the entire set of directions in the super-critical region.

Similar results have been obtained recently in the more general setting of Fortuin–Kasteleyn random cluster models by Bezuidenhout *et al* (1992), using a diffferent technique introduced by Menshikov (1987) and applied by Aizenman and Grimmett (1991) to Ising models.

Amplitudes and amplitude ratios have been investigated via series expansions, Monte Carlo simulation, transfer matrix, and renormalization group methods. Privman *et al* (1991), section 6.5, provides a summary of numerical results for percolation models. The evidence for universality of amplitudes and amplitude ratios in anisotropic models is inconclusive, with Privman *et al* claiming that they need not be identical for different directions, and Nightingale and Blöte (1983) suggesting they are identical for two-dimensional Ising models. The substitution method described in this paper is not sufficiently refined to provide mathematically rigorous proof of universality of amplitudes or amplitude ratios in anisotropic percolation models to settle this question. Note also that, while establishing identity of the critical exponents in different directions in these models, the method does not provide exact values, bounds, or numerical estimates for the exponents.

Acknowledgments

The author gratefully acknowledges participant support from the London Mathematical Society for the Symposium on Probabilistic Methods in Combinatorics in Durham, England, in July, 1991, where the results of this paper were presented.

The author also gratefully acknowledges financial support for this research from the National Science Foundation through grant DMS-9012629.

References

Aizenman M 1987 General results in percolation theory *Proc. Taniguchi Workshop and Symp. on Probabilistic Methods in Mathematical Physics (Kataka and Kyoto)* ed N Ikeda pp 1–22

Aizenman, M and Barsky D J 1987 Sharpness of the phase transition in percolation models *Commun. Math. Phys.* **108** 489–526

Aizenman M and Grimmett G R 1991 Strict monotonicity for critical points in percolation and ferromagnetic models *J. Stat. Phys.* **63** 817–35

Bezuidenhout C E, Grimmett G R and Kesten H 1992 Strict inequality for critical values of Potts models and random cluster processes *Preprint*

Durrett R 1988 *Lecture Notes on Particle Systems and Percolation* (Monterey, CA: Wadsworth and Brooks/Cole)

Efros A L 1986 *Physics and Geometry of Disorder* (Moscow: Mir)

Grimmett G 1989 *Percolation* (Berlin: Springer)

Hammersley J M 1980 A generalization of McDiarmid's theorem for mixed Bernoulli percolation *Proc. Camb. Phil. Soc.* **88** 167–70

Kesten H 1982 *Percolation Theory for Mathematicians.* (Boston: Birkhäuser)

—— 1987 Percolation theory and first-passage percolation *Ann. Prob.* **15** 1231–71

McDiarmid C J H 1980 Clutter percolation and random graphs *Math. Prog. Study* **13** 17–25

—— 1981 General percolation and random graphs *Adv. Appl. Probab.* **13** 40–60

Menshikov M V, Molchanov S A and Sidorenko A F 1986 Percolation theory and some applications *Itogi Nauki i Techniki (Series of Probability Theory, Mathematical Statistics, and Theoretical Cybernetics)* **24** 53–110

Newman C M 1987 Percolation theory: a selective survey of rigorous results *Advances in Multiphase Flow and Related Problems* ed G Papanicolaou 146–67 (Philadelphia: SIAM)

Nightingale P and Blöte H 1983 The relation between amplitudes and critical exponents in finite-size scaling *J. Phys. A: Math. Gen.* **16** L657–64

Preston C J 1974 A generalization of the FKG inequalities *Commun. Math. Phys.* **36** 233–41

Privman V, Hohenberg P C and Aharony A 1991 Universal critical-point amplitude relations *Phase Transitions and Critical Phenomena* vol 14 ed C Domb and J L Lebowitz pp 1–134

Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)

Sykes M F and Essan J W 1964 Exact critical percolation probabilities for site and band problems in two dimensions *J. Math. Phys.* **5** 1117–27

Wierman J C 1984 Mixed percolation on the square lattice *J. Appl. Probab.* **21** 247–59

—— 1990 Bond percolation critical probability bounds for the Kagomé lattice by a substitution method *Disorder in Physical Systems* ed G Grimmett and D J A Welsh (Oxford: Oxford University Press)

—— 1991 Equality of the bond percolation critical exponents for two pairs of dual lattices *Combinatorics, Probability, and Computing* **1** 95–105